CHU-VANDERMONDE CONVOLUTION AND HARMONIC NUMBER IDENTITIES

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Abstract. By applying the derivative operators to Chu-Vandermonde convolution, several general harmonic number identities are established.

1. Introduction

For $l \in \mathbb{N}$ and $n \in \mathbb{N}_0$, define the generalized harmonic numbers by

$$H_0^{\langle l\rangle}=0 \quad \text{and} \quad H_n^{\langle l\rangle}=\sum_{k=1}^{n}\frac{1}{k^l} \quad \text{for} \quad n=1,2,\cdots.$$

When $l=1$, they reduce to the classical harmonic numbers:

$$H_0=0 \quad \text{and} \quad H_n=\sum_{k=1}^{n}\frac{1}{k} \quad \text{for} \quad n=1,2,\cdots.$$

There exist many elegant identities involving harmonic numbers. They can be found in the papers [2], [3], [4], [5], [6], [7], [8], [9] and [10].

For two differentiable functions $f(x)$ and $g(x,y)$, define respectively the derivative operator $\mathcal{D}_x$ and $\mathcal{D}_{xy}^2$ by

$$\mathcal{D}_x f(x)=\frac{d}{dx}f(x)\Big|_{x=0},$$
$$\mathcal{D}_{xy}^2 g(x,y)=\frac{\partial^2}{\partial x\partial y}g(x,y)\Big|_{x=y=0}.$$

Then it is not difficult to show the following two derivatives:

$$\mathcal{D}_x\binom{s+x}{t}=\binom{s}{t}\big\{H_s-H_{s-t}\big\},$$
$$\mathcal{D}_{xy}^2\binom{s+x}{t}\binom{u+y}{v}=\binom{s}{t}\binom{u}{v}\big\{H_s-H_{s-t}\big\}\big\{H_u-H_{u-v}\big\},$$

where $s,t,u,v \in N_0$ with $t \le s$ and $v \le u$.

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There are numerous binomial identities in the literature. Thereinto, Chu-Vandermonde convolution (cf. [1, p. 67]) can be stated as

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k}=\binom{x+y}{n}. \tag{1}$$

By applying the derivative operators $\mathcal{D}_x$ and $\mathcal{D}_{xy}^2$ to (1), several general harmonic number identities will be established in the next section.

2. Harmonic number identities

§2.1. Performing the replacements $x \to -p-1-x$, $y \to -q-1-y$ for (1) with $p,q \in N_0$, we obtain the binomial sum:

$$\sum_{k=0}^{n}\binom{p+k+x}{k}\binom{q+n-k+y}{n-k}=\binom{p+q+n+1+x+y}{n}. \tag{2}$$

Applying the derivative operator $\mathcal{D}_x$ to (2) and then letting $y=0$, we have

$$\sum_{k=0}^{n}\binom{p+k}{p}\binom{q+n-k}{q}H_{p+k}=\binom{p+q+n+1}{n}\Big\{H_p+H_{p+q+n+1}-H_{p+q+1}\Big\}. \tag{3}$$

When $p=q=0$, (3) reduces to the known result (cf. [3, Equation 2.1]):

$$\sum_{k=0}^{n}H_k=(n+1)\Big\{H_{n+1}-1\Big\}.$$

Employing the substitutions $p \to p+1$, $k \to k-1$, $n \to n-1$ for (3), we have

$$\begin{aligned}\sum_{k=1}^{n}\binom{p+k}{k}\binom{q+n-k}{q}kH_{p+k} &= (p+1)\binom{p+q+n+1}{n-1}\\ &\quad\times\Big\{H_{p+1}+H_{p+q+n+1}-H_{p+q+2}\Big\}.\end{aligned} \tag{4}$$

When $p=q=0$, (4) reduces to the known result (cf. [3, Equation 2.2]):

$$\sum_{k=1}^{n}kH_k=\frac{n(n+1)}{2}H_n-\frac{(n-1)n}{4}.$$

The method, which is used to deduce (4), can further be utilized to derive equations with the factor k^m, where $m \geq 2$. Considering that the general resulting identities will become complicated, we shall only recover the two known results (cf. [3, Equations 2.3-2.4]):

$$\begin{aligned}\sum_{k=1}^{n}k^2H_k &= \frac{n(n+1)(2n+1)}{6}H_n-\frac{(n-1)n(4n+1)}{36},\\ \sum_{k=1}^{n}k^3H_k &= \frac{n^2(n+1)^2}{4}H_n-\frac{(n-1)n(n+1)(3n-2)}{48}.\end{aligned}$$

Applying the derivative operator $\mathcal{D}_{xy}^2$ to (2) and using (3), we establish the theorem.

Theorem 1. *For $p,q \in N_0$, there holds the harmonic number identity:*

$$\begin{aligned}\sum_{k=0}^{n}\binom{p+k}{p}\binom{q+n-k}{q}H_{p+k}H_{q+n-k} &= \binom{p+q+n+1}{n}\Big\{\big(H_{p+q+1}^{\langle 2\rangle}-H_{p+q+n+1}^{\langle 2\rangle}\big)\\ &\quad+\big(H_p-H_{p+q+1}+H_{p+q+n+1}\big)\big(H_q-H_{p+q+1}+H_{p+q+n+1}\big)\Big\}.\end{aligned}$$

When $p = q = 0$, Theorem 1 reduces to the identity:

$$\sum_{k=0}^{n} H_k H_{n-k} = (n+1)\Big\{\big(1 - H_{n+1}^{\langle 2\rangle}\big) + \big(H_{n+1} - 1\big)^2\Big\}.$$

Employing the substitutions $p \to p+1$, $k \to k-1$, $n \to n-1$ for Theorem 1, we found the theorem.

Theorem 2. *For $p, q \in N_0$, there holds the harmonic number identity:*

$$\sum_{k=1}^{n} \binom{p+k}{p}\binom{q+n-k}{q} k H_{p+k} H_{q+n-k} = \binom{p+q+n+1}{n-1}\Big\{\big(H_{p+q+2}^{\langle 2\rangle} - H_{p+q+n+1}^{\langle 2\rangle}\big)$$
$$+\big(H_{p+1} - H_{p+q+2} + H_{p+q+n+1}\big)\big(H_q - H_{p+q+2} + H_{p+q+n+1}\big)\Big\}(p+1).$$

When $p = q = 0$, Theorem 2 reduces to the identity:

$$\sum_{k=1}^{n} k H_k H_{n-k} = \frac{n(n+1)}{2}\Big\{H_{n+1}^2 - H_{n+1}^{\langle 2\rangle} - 2H_{n+1} + 2\Big\}.$$

Further, we can deduce the following two identities:

$$\sum_{k=1}^{n} k^2 H_k H_{n-k} = \frac{n(n+1)(2n+1)}{6}\Big\{H_{n+1}^2 - H_{n+1}^{\langle 2\rangle} - \frac{13n+5}{3(2n+1)}H_{n+1} + \frac{71n+37}{18(2n+1)}\Big\},$$
$$\sum_{k=1}^{n} k^3 H_k H_{n-k} = \frac{n^2(n+1)^2}{4}\Big\{H_{n+1}^2 - H_{n+1}^{\langle 2\rangle} - \frac{7n+5}{3(n+1)}H_{n+1} + \frac{35n+37}{18(n+1)}\Big\}.$$

§2.2. Performing the replacements $x \to p+n+x$, $y \to q+n+y$ for (1) with $p, q \in N_0$, we get the binomial sum:

$$\sum_{k=0}^{n} \binom{p+n+x}{k}\binom{q+n+y}{n-k} = \binom{p+q+2n+x+y}{n}. \tag{5}$$

Applying the derivative operator $\mathcal{D}_y$ to (5) and then letting $x = 0$, we have

$$\sum_{k=0}^{n} \binom{p+n}{k}\binom{q+n}{n-k} H_{q+k} = \binom{p+q+2n}{n}\Big\{H_{q+n} + H_{p+q+n} - H_{p+q+2n}\Big\}, \tag{6}$$

which is a special case of [4, Theorem 1.5].

Applying the derivative operator $\mathcal{D}_{xy}^2$ to (5) and using (6), we establish the theorem.

Theorem 3. *For $p, q \in N_0$, there holds the harmonic number identity:*

$$\sum_{k=0}^{n} \binom{p+n}{k}\binom{q+n}{n-k} H_{p+n-k} H_{q+k} = \binom{p+q+2n}{n}\Big\{\big(H_{p+q+n}^{\langle 2\rangle} - H_{p+q+2n}^{\langle 2\rangle}\big)$$
$$+\big(H_{p+n} + H_{p+q+n} - H_{p+q+2n}\big)\big(H_{q+n} + H_{p+q+n} - H_{p+q+2n}\big)\Big\}.$$

When $p = q = 0$, Theorem 3 reduces to the known result due to Chen and Chu [5, Example 3]:

$$\sum_{k=0}^{n} \binom{n}{k}^2 H_k H_{n-k} = \binom{2n}{n}\Big\{\big(H_n^{\langle 2\rangle} - H_{2n}^{\langle 2\rangle}\big) + \big(2H_n - H_{2n}\big)^2\Big\}.$$

Employing the substitutions $q \to q+1$, $k \to k-1$, $n \to n-1$ for Theorem 3, we found the theorem.

Theorem 4. *For $p,q\in N_0$, there holds the harmonic number identity:*

$$\sum_{k=1}^{n}\binom{p+n}{k}\binom{q+n}{n-k}kH_{p+n-k}H_{q+k}=\binom{p+q+2n-1}{n-1}\Big\{\big(H^{\langle 2\rangle}_{p+q+n}-H^{\langle 2\rangle}_{p+q+2n-1}\big)$$
$$+\big(H_{p+n-1}+H_{p+q+n}-H_{p+q+2n-1}\big)\big(H_{q+n}+H_{p+q+n}-H_{p+q+2n-1}\big)\Big\}(p+n).$$

When $p=q=0$, Theorem 4 reduces to the identity:

$$\begin{aligned}\sum_{k=1}^{n}\binom{n}{k}^2 kH_kH_{n-k} &= \frac{n}{2}\binom{2n}{n}\Big\{\big(H^{\langle 2\rangle}_{n}-H^{\langle 2\rangle}_{2n-1}\big)\\ &\quad+\big(2H_n-H_{2n-1}\big)\big(2H_n-H_{2n-1}-1/n\big)\Big\}.\end{aligned}$$

Further, we can derive the following two identities:

$$\sum_{k=1}^{n}\binom{n}{k}^2 k^2H_kH_{n-k}=\frac{n^3}{4n-2}\binom{2n}{n}\bigg\{\Big(H^{\langle 2\rangle}_{n}-H^{\langle 2\rangle}_{2n-1}\Big)$$
$$+\Big(2H_n-H_{2n-1}\Big)\Big(2H_n-H_{2n-1}-\frac{2n^2-1}{2n^3-n^2}\Big)-\frac{(n-1)(2n^2-2n+1)}{n^3(2n-1)^2}\bigg\},$$
$$\sum_{k=1}^{n}\binom{n}{k}^2 k^3H_kH_{n-k}=\frac{n^3(n+1)}{8n-4}\binom{2n}{n}\bigg\{\Big(H^{\langle 2\rangle}_{n}-H^{\langle 2\rangle}_{2n-1}\Big)$$
$$+\Big(2H_n-H_{2n-1}\Big)\Big(2H_n-H_{2n-1}-\frac{2n^2+4n-4}{2n^3+n^2-n}\Big)-\frac{3(n-1)(2n^2-2n+1)}{n^2(n+1)(2n-1)^2}\bigg\}.$$

§2.3. Performing the replacements $x\to -p-1-x$, $y\to q+n+y$ for (1) with $p,q\in N_0$, we achieve the binomial sum:

$$\sum_{k=0}^{n}(-1)^k\binom{p+k+x}{k}\binom{q+n+y}{n-k}=(-1)^n\binom{p-q+x-y}{n}. \tag{7}$$

Applying the derivative operator $\mathcal{D}_x$ to (7) and then letting $y=0$, we have

$$\sum_{k=0}^{n}(-1)^k\binom{p+k}{k}\binom{q+n}{n-k}H_{p+k}=\begin{cases}A_n, & \text{for } p-q\geq n,\\ B_n, & \text{for } 0\leq p-q<n,\\ C_n, & \text{for } p-q<0,\end{cases} \tag{8}$$

where

$$\begin{aligned}A_n&=(-1)^n\binom{p-q}{n}\big\{H_p+H_{p-q}-H_{p-q-n}\big\},\\ B_n&=(-1)^{1+p-q}\frac{(p-q)!(q-p+n-1)!}{n!},\\ C_n&=(-1)^n\binom{p-q}{n}\big\{H_p+H_{q-p-1}-H_{q-p+n-1}\big\}.\end{aligned}$$

Applying the derivative operator $\mathcal{D}_y$ to (7) and then letting $x=0$, we have

$$\sum_{k=0}^{n}(-1)^k\binom{p+k}{k}\binom{q+n}{n-k}H_{q+k}=\begin{cases}D_n, & \text{for } p-q\geq n,\\ E_n, & \text{for } 0\leq p-q<n,\\ F_n, & \text{for } p-q<0,\end{cases} \tag{9}$$

where

$$\begin{aligned}
D_n &= (-1)^n\binom{p-q}{n}\big\{H_{q+n}+H_{p-q}-H_{p-q-n}\big\},\\
E_n &= (-1)^{1+p-q}\frac{(p-q)!(q-p+n-1)!}{n!},\\
F_n &= (-1)^n\binom{p-q}{n}\big\{H_{q+n}+H_{q-p-1}-H_{q-p+n-1}\big\}.
\end{aligned}$$

We point out that (8) can be given by [4, Theorem 1.1] and (9) can be offered by [4, Theorem 1.5].

Applying the derivative operator $\mathcal{D}^2_{xy}$ to (7) and using (8)-(9), we establish the theorem.

Theorem 5. *For $p,q \in N_0$, there holds the harmonic number identity:*

$$\sum_{k=0}^{n}(-1)^k\binom{p+k}{k}\binom{q+n}{n-k}H_{p+k}H_{q+k}=\begin{cases}U_n, \text{ for } p-q\geq n,\\ V_n, \text{ for } 0\leq p-q<n,\\ W_n, \text{ for } p-q<0,\end{cases}$$

where

$$\begin{aligned}
U_n \;&=\; (-1)^n\binom{p-q}{n}\Big\{\big(H^{\langle 2\rangle}_{p-q-n}-H^{\langle 2\rangle}_{p-q}\big)\\
&\quad+\big(H_p+H_{p-q}-H_{p-q-n}\big)\big(H_{q+n}+H_{p-q}-H_{p-q-n}\big)\Big\},\\
V_n \;&=\; (-1)^{1+p-q}\frac{(p-q)!(q-p+n-1)!}{n!}\\
&\quad\times\Big\{H_p+H_{q+n}+2H_{p-q}-2H_{q-p+n}+\frac{2}{q-p+n}\Big\},\\
W_n \;&=\; (-1)^n\binom{p-q}{n}\Big\{\big(H^{\langle 2\rangle}_{q-p-1}-H^{\langle 2\rangle}_{q-p+n-1}\big)\\
&\quad+\big(H_p+H_{q-p-1}-H_{q-p+n-1}\big)\big(H_{q+n}+H_{q-p-1}-H_{q-p+n-1}\big)\Big\}.
\end{aligned}$$

When $p=q=0$ with $n>0$, Theorem 5 reduces to the identity:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_k^2=\frac{1}{n}\Big\{H_n-\frac{2}{n}\Big\}.$$

Employing the substitutions $p\to p+1$, $q\to q+1$, $k\to k-1$, $n\to n-1$ for Theorem 5, we found the theorem.

Theorem 6. *For $p,q \in N_0$, there holds the harmonic number identity:*

$$\sum_{k=1}^{n}(-1)^k\binom{p+k}{k}\binom{q+n}{n-k}kH_{p+k}H_{q+k}=\begin{cases}U_n^*, \text{ for } p-q\geq n-1,\\ V_n^*, \text{ for } 0\leq p-q<n-1,\\ W_n^*, \text{ for } p-q<0,\end{cases}$$

where

$$
\begin{aligned}
U_n^* &= (-1)^n(p+1)\binom{p-q}{n-1}\Big\{\big(H^{\langle 2\rangle}_{p-q-n+1}-H^{\langle 2\rangle}_{p-q}\big)\\
&\quad+\big(H_{p+1}+H_{p-q}-H_{p-q-n+1}\big)\big(H_{q+n}+H_{p-q}-H_{p-q-n+1}\big)\Big\},\\
V_n^* &= (-1)^{p-q}(p+1)\frac{(p-q)!(q-p+n-2)!}{(n-1)!}\\
&\quad\times\Big\{H_{p+1}+H_{q+n}+2H_{p-q}-2H_{q-p+n-1}+\frac{2}{q-p+n-1}\Big\},\\
W_n^* &= (-1)^n(p+1)\binom{p-q}{n-1}\Big\{\big(H^{\langle 2\rangle}_{q-p-1}-H^{\langle 2\rangle}_{q-p+n-2}\big)\\
&\quad+\big(H_{p+1}+H_{q-p-1}-H_{q-p+n-2}\big)\big(H_{q+n}+H_{q-p-1}-H_{q-p+n-2}\big)\Big\}.
\end{aligned}
$$

When $p=q=0$ with $n>1$, Theorem 6 reduces to the identity:

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}kH_k^2=\frac{1}{1-n}\Big\{H_n-\frac{n^2+3n-2}{n(n-1)}\Big\}.$$

Further, we can deduce the following two identities:

$$
\begin{aligned}
\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^2H_k^2 &= \frac{n}{(n-1)(n-2)}\Big\{H_n-\frac{2n^3+n^2-11n+6}{n(n-1)(n-2)}\Big\},\\
\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^3H_k^2 &= \frac{(n+1)n}{(n-1)(n-2)(n-3)}\\
&\qquad\times\Big\{H_n-\frac{(3n^4-4n^3-32n^2+62n-15)n-6}{(n+1)n(n-1)(n-2)(n-3)}\Big\},
\end{aligned}
$$

where $n>2$ for the first equation and $n>3$ for the second equation.

Remark: With the change of the parameters p and q, Theorems 1-6 can produce more interesting harmonic number identities. We shall not lay them out one by one because of the triviality of the work.

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